

Cellular sheaf cohomology in Polymake

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Abstract

This chapter provides a guide to our *polymake* extension **cellularSheaves**. We first define cellular sheaves on polyhedral complexes in Euclidean space, as well as cosheaves, and their (co)homologies. As motivation, we summarise some results from toric and tropical geometry linking cellular sheaf cohomologies to cohomologies of algebraic varieties. We then give an overview of the structure of the extension **cellularSheaves** for *polymake*. Finally, we illustrate the usage of the extension with examples from toric and tropical geometry.

1 Introduction

Given a polyhedral complex Π in \mathbb{R}^n , a cellular sheaf (of vector spaces) on Π associates to every face of Π a vector space and to every face relation a map of the associated vector spaces (see Definition 2). Just as with usual sheaves, we can compute the cohomology of cellular sheaves (see Definition 5). The advantage is that cellular sheaf cohomology is the cohomology of a chain complex consisting of finite dimensional vector spaces (see Definition 3).

Polyhedral complexes often arise in algebraic geometry. Moreover, in some cases, invariants of algebraic varieties can be recovered as cohomology of certain cellular sheaves on polyhedral complexes. Our two major classes of examples come from toric and tropical geometry and are presented in Section 2.2. We refer the reader to [9] for a guide to toric geometry and polytopes. For an introduction to tropical geometry see [4], [20].

The main motivation for this *polymake* extension was to implement tropical homology, as introduced by Itenberg, Katzarkov, Mikhalkin, and Zharkov in [15]. Tropical homology is the homology of particular cosheaves which can be defined on any polyhedral complex. When the polyhedral complex arises as the tropicalisation of a family of complex projective varieties, the tropical homology groups give information about the Hodge numbers of a generic member of the family. However, this is just one particular instance of cellular (co)sheaf (co)homology which our package is capable of dealing with. Cellular (co)sheaves have also been a powerful tool in recent years in the field of applied topology, notably in persistent homology, sensor networks and network coding [11], [5]. With this *polymake* extension, (co)sheaves on polyhedral complexes can be constructed from scratch. We

hope that this will allow for a range of uses of the extension beyond just the ones from combinatorial algebraic geometry that we point out here.

The definitions of cellular (co)sheaves and their (co)homologies are given in Section 2. Then a description of our implementation of cellular sheaves and their cohomologies in *polymake* is given in Section 3. The framework *polymake* already provides a large number of combinatorial objects and tools, making it easy to construct the polyhedral complexes of interest. In Section 4, we illustrate the usage of the extension in a variety of examples from tropical and algebraic geometry. Finally, Section 5 outlines some potential future directions and applications for our extension.

2 Cellular sheaf cohomology

We begin by defining cellular sheaves and cosheaves, as well as their cohomologies and homologies. We provide an explicit example of a sheaf and a cosheaf which have been implemented in our package. In Section 2.2 we give an overview of some theorems relating these sheaf cohomologies and cosheaf homologies to the cohomology of some algebraic varieties.

2.1 Definitions

A *polyhedral complex* Π is a finite collection of polyhedra in \mathbb{R}^n with the property that any face of a polyhedron in Π is also in Π and the intersection of any two polyhedra in Π is a face of both.

Let Π^i denote the collection of polyhedra in Π of dimension i . For polyhedra $\sigma, \tau \in \Pi$, we use the notation $\tau \leq \sigma$ to denote that τ is a face of σ .

Definition 1. Given a polyhedral complex Π and a chosen orientation of each polyhedron in Π define the *orientation map* for each i ,

$$\mathcal{O} : \Pi^{i-1} \times \Pi^i \rightarrow \{0, -1, +1\}$$

by

$$\mathcal{O}(\tau, \sigma) := \begin{cases} 0 & \text{if } \tau \not\leq \sigma \\ +1 & \text{if the orientation of } \tau \subset \partial\sigma \text{ coincides with that} \\ & \text{of } \tau \\ -1 & \text{if the orientation of } \tau \subset \partial\sigma \text{ differs from that of} \\ & \tau. \end{cases}$$

A polyhedral complex Π can be considered as a category where the objects are the polyhedra and the morphisms are given by inclusions. i.e. $(f : \tau \rightarrow \sigma) \in \text{Mor}(\Pi)$ if and only if $\tau \leq \sigma$. We use the notation Π^{op} to denote the category obtained from Π by using the same objects and reversing the directions of all morphisms.

We will be interested in cellular sheaves and cosheaves of vector spaces on polyhedral complexes. Viewing Π as a category as described above, we can give a succinct definition of cellular (co)sheaves. Let Vect_K denote the category of vector spaces over a field K .

Definition 2. Let Π be a polyhedral complex, then a *cellular sheaf* \mathcal{G} and a *cellular cosheaf* \mathcal{F} are functors

$$\mathcal{G} : \Pi \rightarrow \text{Vect}_K \quad \mathcal{F} : \Pi^{op} \rightarrow \text{Vect}_K.$$

To summarise, this means that a cellular sheaf consists of the following data:

- for each polyhedron σ in Π a vector space $\mathcal{G}(\sigma)$ and,
- given $\tau, \sigma \in \Pi$ satisfying $\tau \leq \sigma$, a morphism $\rho_{\tau\sigma} : \mathcal{G}(\tau) \rightarrow \mathcal{G}(\sigma)$.

In particular, for $\gamma \leq \tau \leq \sigma$ the restriction morphisms commute, i.e. we have

$$\rho_{\gamma\sigma} = \rho_{\tau\sigma} \circ \rho_{\gamma\tau}.$$

A cellular cosheaf is similar except that the morphisms are in the opposite direction $\iota_{\sigma\tau} : \mathcal{F}(\sigma) \rightarrow \mathcal{F}(\tau)$.

A sheaf of vector spaces in the usual sense is a contravariant functor from the category of open sets of a topological space to Vect_K which satisfies additional axioms. A polyhedral complex can be equipped with a finite topology known as the Alexandrov topology, and the above definition of cellular sheaf as a functor produces a sheaf in the usual sense in this topology. Due to the simplicity of the cellular sheaves and the Alexandrov topology, no additional sheaf axioms are required. The reader is directed to [5, Chapter 4] for more details.

Example 1. As a first example we can define the constant sheaf by setting $\mathcal{G}(\sigma)$ to be the one dimensional vector space K and the maps $\rho_{\tau\sigma} : \mathcal{G}(\tau) \rightarrow \mathcal{G}(\sigma)$ to be the identity for all $\tau, \sigma \in \Pi$ such that $\tau \leq \sigma$. The constant cosheaf can be defined in a similar fashion.

Example 2. Let Π be a polyhedral complex in \mathbb{R}^n . For $\sigma \in \Pi$ set $L(\sigma)$ to be the linear subspace of \mathbb{R}^n parallel to the face σ . For $p \in \mathbb{Z}_{\geq 0}$ we define the sheaf W^p as follows:

$$W^p(\sigma) = \wedge^p L(\sigma)$$

and the maps $\rho_{\tau\sigma} : W^p(\tau) \rightarrow W^p(\sigma)$ are given by the wedges of the natural inclusions $L(\tau) \rightarrow L(\sigma)$ for $\tau \leq \sigma$. By convention, the sheaf W^0 is the constant sheaf from Example 1.

Example 3. Next we give an example of a cosheaf on a polyhedral complex. The homology of this particular cosheaf is the tropical homology from [15], and will come up in many examples in Sections 4.2 and 4.3.

Let Π be a polyhedral complex in \mathbb{R}^n , then we define

$$F_p(\sigma) = \sum_{\sigma < \gamma} \wedge^p L(\gamma).$$

If $\tau \leq \sigma$, then $\{\gamma \mid \sigma < \gamma\} \subset \{\gamma \mid \tau < \gamma\}$, so we get a natural inclusion $\iota_{\sigma\tau} : F_p(\sigma) \rightarrow F_p(\tau)$. Analogously to Example 2, we obtain the constant cosheaf from Example 1 for $p = 0$.

Note that by dualising the vector spaces $\mathcal{G}(\sigma)$ for all σ we can transform a cellular sheaf \mathcal{G} into a cellular cosheaf in a natural way. A sheaf can be created from a cosheaf in the analogous way.

For a given (co)sheaf we build (co)chain complexes in the following two parallel definitions. The definitions originally appeared in this form in [5].

Definition 3. Given a polyhedral complex Π and a cellular sheaf \mathcal{G} , define the *cellular cochain groups* and *cellular cochain groups with compact support*, respectively, as

$$C^q(\Pi; \mathcal{G}) := \bigoplus_{\substack{\dim \sigma = q \\ \sigma \text{ compact}}} \mathcal{G}(\sigma) \quad \text{and} \quad C_c^q(\Pi; \mathcal{G}) := \bigoplus_{\dim \sigma = q} \mathcal{G}(\sigma).$$

The cellular cochain maps (usual or with compact support)

$$d : C^q(\Pi; \mathcal{G}) \rightarrow C^{q+1}(\Pi; \mathcal{G}) \quad \text{and} \quad d : C_c^q(\Pi; \mathcal{G}) \rightarrow C_c^{q+1}(\Pi; \mathcal{G})$$

are given component-wise for $\tau \in \Pi^q$ and $\sigma \in \Pi^{q+1}$ by $d_{\tau\sigma} : \mathcal{G}(\tau) \rightarrow \mathcal{G}(\sigma)$, where

$$d_{\tau\sigma} := \begin{cases} \mathcal{O}(\tau, \sigma) \cdot \rho_{\tau\sigma} & \tau \leq \sigma \\ 0 & \text{else.} \end{cases}$$

Definition 4. Given a polyhedral complex Π and a cellular cosheaf \mathcal{F} define the *cellular chain groups* and the *Borel-Moore cellular chain groups*, respectively, as

$$C_q(\Pi; \mathcal{F}) := \bigoplus_{\substack{\dim \sigma = q \\ \sigma \text{ compact}}} \mathcal{F}(\sigma) \quad \text{and} \quad C_q^{BM}(\Pi; \mathcal{F}) := \bigoplus_{\dim \sigma = q} \mathcal{F}(\sigma).$$

The cellular chain maps (usual or Borel-Moore)

$$\partial : C_q(\Pi; \mathcal{F}) \rightarrow C_{q-1}(\Pi; \mathcal{F}) \quad \text{and} \quad \partial : C_q^{BM}(\Pi; \mathcal{F}) \rightarrow C_{q-1}^{BM}(\Pi; \mathcal{F})$$

are given component-wise for $\sigma \in \Pi^q$ and $\tau \in \Pi^{q-1}$ by $\partial_{\sigma\tau} : \mathcal{F}(\sigma) \rightarrow \mathcal{F}(\tau)$, where

$$\partial_{\sigma\tau} := \begin{cases} \mathcal{O}(\sigma, \tau) \cdot \iota_{\sigma\tau} & \sigma \geq \tau \\ 0 & \text{else.} \end{cases}$$

Remark 1. It may seem counter intuitive that the usual cellular cochains are supported only on compact faces and cellular cochains with compact support are supported on all faces. After we define cohomology of a cellular sheaf and homology of a cellular cosheaf below, a sanity check can be performed to compute the cohomology of the constant sheaf from Example 1 on your favourite non-compact polyhedral complex. Under some reasonable conditions on the polyhedral complex, the cellular cohomology of the constant sheaf will be isomorphic to the ordinary singular cohomology of the polyhedral complex. The analogous statement holds for the compactly supported versions. See [5, Example 6.2.4] for a simple example and more details.

Definition 5. The cellular sheaf cohomology (with compact support) of \mathcal{G} is the cohomology of the cellular cochain complex (with compact support) from Definition 3.

The cellular (Borel-Moore) cosheaf homology of \mathcal{F} is the homology of the cellular (Borel-Moore) chain complex from Definition 4.

2.2 Connections with classical algebraic geometry

In this section we present some particular connections between cellular (co)homologies of certain (co)sheaves and cohomology of complex algebraic varieties. Explicit demonstrations of the statements of these theorems are given along with the *polymake* code in Section 4.

To a rational polytope $\Delta \subset \mathbb{R}^n$ we can associate a toric variety $TV(\Delta)$ by first considering its normal fan then building the toric variety corresponding to this fan following [9]. The following theorem relates the cohomology of the sheaves of p -differential forms Ω^p on $TV(\Delta)$ with the cohomologies of the sheaves W^p from Example 2 on the polytope Δ .

Theorem 1. [7, Remark 12.4.1] *Let $\Delta \subset \mathbb{R}^n$ be a rational polytope and $TV(\Delta)$ its associated toric variety. Then*

$$H^q(TV(\Delta); \Omega^p) \cong H^q(\Delta; W^p) \otimes_{\mathbb{R}} \mathbb{C}.$$

In particular, when the rational polytope Δ is simple, the associated toric variety is smooth and $H^q(TV(\Delta); \Omega^p) \cong H^{p,q}(TV(\Delta))$, where the last vector space denotes the $(p, q)^{\text{th}}$ part of the Hodge decomposition of the cohomology of $TV(\Delta)$. As an example in Section 4.1, we compute $H^q(\Delta; W^p)$ for Δ a three dimensional cube using the extension **cellularSheaves**.

The next two connections relate to the F -cosheaves defined in Example 3. We begin with an arrangement of hyperplanes \mathcal{A} in $\mathbb{C}P^d$. It is a theorem of Orlik and Solomon that the cohomology of the complement of the arrangement depends only on the combinatorics of the arrangement, in other words the corresponding matroid (see [18] or [22] for an introduction to matroid theory). Moreover, for any matroid there is a combinatorially described Orlik-Solomon algebra, which provides the cohomology ring of the complement when we are in the situation above (see [21]).

To any matroid M we can associate a fan $B(M)$ in Euclidean space known as the Bergman fan of M (see [1]). If M is the matroid of an arrangement \mathcal{A} the fan $B(M)$ is the tropicalisation of the complement $\mathbb{C}P^d \setminus \mathcal{A}$ under a suitable embedding to a complex torus.

Theorem 2 ([26]). *Let M be any matroid, $B(M)$ its Bergman fan, and F_p the cosheaves from Example 3 on $B(M)$, then*

$$OS^p(M) = F_p(v)^*,$$

where OS^p denotes the p^{th} graded part of the Orlik-Solomon algebra of M , and v is the vertex of the fan.

In Section 4.2, we illustrate how we can compute the dimensions of the graded pieces of the Orlik-Solomon algebra of a matroid using the *polymake* applications **matroid**, **tropical** and the extension **cellularSheaves**.

Lastly, the statements relating the cohomology of the complements of arrangements and the tropical homology of matroidal fans in Theorem 2 can be generalised and even

refined in the setting of tropicalisation of projective complex algebraic varieties. We state the theorem and refer the reader to [15] for the precise definitions of smooth \mathbb{Q} -tropical projective varieties and tropical limits.

Theorem 3. [15] *Consider a 1-parameter family of complex projective varieties $\pi : \mathcal{X} \rightarrow D^\circ$, where D° is the punctured disc. Suppose that the tropical limit $\text{Trop}(\mathcal{X}) = X$ is a smooth \mathbb{Q} -tropical projective variety. Then*

$$\dim H^{p,q}(\mathcal{X}_t) = \dim H_q(X; F_p),$$

where $H^{p,q}(\mathcal{X}_t)$ is the $(p, q)^{\text{th}}$ part of the Hodge decomposition of $\mathcal{X}_t = \pi^{-1}(t)$ for a generic $t \in D^\circ$.

It is due to the above theorem that the homology of the F -cosheaves is also known as tropical homology.

As of yet, *polymake* does not have compact tropical varieties as objects. Therefore, in the examples presented in Section 4, we do not produce Hodge numbers of complex projective algebraic varieties, but rather Betti numbers of limit mixed Hodge structures of non-complete varieties. Examples of this can be found in [6] for hypersurfaces and complete intersections, also in [19] from a more tropical point of view. Future plans to implement tropical homology for compact (and hence also projective) tropical varieties are outlined in Section 5.2.

3 Implementation in *polymake*

The framework *polymake* is a mathematical software for polyhedral geometry. Its objects of interest are mainly combinatorial, such as cones, polyhedra, graphs, fans and polyhedral complexes. Toric and tropical geometry provide many ways for using *polymake* to solve computational tasks from algebraic geometry. In particular, *polymake* provides the applications `tropical` for tropical geometry and `fulton` for toric geometry. Furthermore, *polymake* interfaces several other software packages which may be useful in our context, such as *gfan* ([17]) for tropical computations and *Singular* ([8]) for algebraic geometry. See [13] for an overview of the most current implemented *polymake* features for tropical geometry.

The interface language of *polymake* is perl. For improved performance one can write and attach C++ code. The combinatorial objects are realised as objects with properties, e.g. the object `Polytope` has the properties `VERTICES` and `F_VECTOR` amongst many others. Since solving certain problems can be very expensive time- and resource-wise, *polymake* adheres to the principle of lazy evaluation: properties are only computed when needed and then stored with the object, so they do not have to be recomputed.

Computation of properties is done via *polymake*'s internal rule structure. A rule takes a certain set of input properties and then computes a certain set of output properties. When asked for a certain property of an object, *polymake* creates a queue of rules to apply in order to get this property from any set of given properties, if this is possible.

Take for example the following code snippet:

```

object PolyhedralFan {

  property ORIENTATIONS : Map<Set<Set<Int> >, Int>;

  rule ORIENTATIONS: HASSE_DIAGRAM, FAN_DIM, RAYS,
    LINEALITY_SPACE{
... # Code
  }

}

```

Here the object `PolyhedralFan` is equipped with a new property `ORIENTATIONS` which one needs for computing tropical homology, see Definition 1. Then a rule is created, that computes `ORIENTATIONS` from the properties `HASSE_DIAGRAM`, `FAN_DIM`, `RAYS` and `LINEALITY_SPACE` of the `PolyhedralFan`.

Internally in *polymake*, every polyhedral complex Π in \mathbb{R}^n is considered as a polyhedral fan Σ in \mathbb{R}^{n+1} intersected with the hyperplane defined by $x_0 = 1$. Every face of Π is indexed by a subset of the rays of Σ . The one dimensional faces of Σ whose direction $\vec{v} = (v_0, \dots, v_n)$ satisfies $v_0 = 1$ correspond to vertices of Π . The one dimensional faces of Σ whose direction \vec{v} satisfies $v_0 = 0$ correspond to unbounded one dimensional faces of Π .

Definition 6. Let Σ be a fan in \mathbb{R}^{n+1} and $\Pi = \Sigma \cap \{x_0 = 1\}$. A far vertex of Π is a ray of Σ whose direction \vec{v} satisfies $v_0 = 0$. A face σ of Π is,

- a far face if its index set consists only of far vertices;
- a non far face if its index set contains at least one non-far vertex;
- a bounded face if it contains no far vertices;
- an unbounded face if it is neither a far face nor a bounded face.

Our extension **cellularSheaves** adds the properties `FAR_FACES`, `BOUNDED_FACES`, `UNBOUNDED_FACES` to a polyhedral complex.

Computing orientations for the polyhedral fan avoids complications caused by the different types of faces of the polyhedral complex. Since the object `PolyhedralComplex` is derived from the object `PolyhedralFan`, it will have the property `ORIENTATIONS` as well.

3.1 Obtaining the cellularSheaves extension

The extension can be installed on a Linux system with the most recent *polymake* version with the following two steps. First clone the repository with

```

git clone \
  \protect\vrule width0pt\protect\href{http://www.github.com/lkastner/cellularSheaves}\{
  FOLDER

```

into a folder named `FOLDER`. Second start *polymake*, and import the extension using

```
import_extension("FOLDER");
```

The extension introduces the new objects `Sheaf` and `CoSheaf` from Definition 2. See Section 3.2 for more details on the implementation. A basic usage scenario looks like

```
application "fan";
$pc = new PolyhedralComplex(
    check_fan_objects(new Cone(cube(3))));
$w1 = $pc->wsheaf(1);
```

First we switch to the application `fan`, since this is the application our extension adds functionality to. The next line takes the three dimensional cube and turns it into a polyhedral complex. Then we ask for the W^1 -sheaf of Example 2.

We implemented most methods dealing with pure linear algebra in C++. The file

```
apps/fan/include/linalg.h
```

contains the C++ code. These linear algebra methods, especially those assembling a chain complex from given block matrices, perform significantly better when implemented in C++ than in perl.

3.2 Sheaves and cosheaves

In our extension we introduce the objects `Sheaf` and `CoSheaf`. As implemented, these objects have two properties. The first is a map from a collection of sets of integers to matrices. This property represents the vector spaces of a (co)sheaf. The second is a map from pairs of sets in this collection to matrices. These matrices represent the morphisms between these vector spaces.

The vector spaces and morphisms are stored in the following two properties of a (co)sheaf:

```
property BASES : Map<Set<Int>, Matrix>;
```

```
property BLOCKS : Map<Set<Set<Int> >, Matrix >;
```

Let us rephrase this in terms of the Definitions 3 and 4. Let Π be a polyhedral complex with a sheaf \mathcal{G} , and let $\tau \leq \sigma$ be a face relation in Π . The faces of Π are encoded as index sets of the rays of vertices of the defining polyhedral fan Σ . As an object in *polymake*, the sheaf \mathcal{G} has the property `BASES` containing the bases of the vector spaces $\mathcal{G}(\gamma)$ for all γ in Π . Also for the sheaf \mathcal{G} , the property `BLOCKS` contains a matrix representing the map $\rho_{\tau\sigma} : \mathcal{G}(\tau) \rightarrow \mathcal{G}(\sigma)$, for each pair of faces $\tau \leq \sigma$. This matrix is written using the bases from the property `BASES`. Analogously for cosheaves, the property `BASES` of \mathcal{F} contains the bases of $\mathcal{F}(\tau)$ for all $\tau \in \Pi$. The property `BLOCKS` contains a matrix representing the map $\iota_{\sigma\tau} : \mathcal{F}(\sigma) \rightarrow \mathcal{F}(\tau)$, for each pair of faces $\tau \leq \sigma$.

For the purpose of computing sheaf cohomology it is also necessary to store morphisms for certain non-face relations, these will then consist of zero matrices of the appropriate sizes.

The main (co)sheaf constructors are `fcosheaf`, `wsheaf` which produce the (cosheaves) from Examples 2 and 3. These are user methods attached to a polyhedral complex. Each of these methods takes a non-negative integer as parameter, which determines the p in the wedge power for the F -cosheaves and W -sheaves.

3.3 Chain complexes and homologies

The last new important objects are chain complexes introduced as `ChainComplex`. A chain complex comes with the properties `DIFFERENTIALS`, `BETTI_NUMBERS`, `HOMOLOGY` and `IS_WELLDEFINED`. It can be created by giving an array of matrices as the property `INPUT_DIFFERENTIALS`. Furthermore it has a user method `print()` providing a human readable sequence format of the chain complex. Dually, we introduce the object `CoChainComplex`. In reality this is just a wrapper around the object `ChainComplex` for the user's convenience.

Currently there are two (co)homology methods in our extension for a given (co)sheaf. They differ by which faces are considered when building the chain complex.

1. `usual_chain_complex`: This method considers only the bounded faces of the given polyhedral complex, i.e. it computes $C_{\bullet}(\Pi; \mathcal{F})$.
2. `borel_moore_complex`: This method uses all non-far faces of a given polyhedral complex, i.e. it computes $C_{\bullet}^{BM}(\Pi; \mathcal{F})$.

Analogously

3. `usual_cochain_complex` gives $C^{\bullet}(\Pi; \mathcal{G})$ and
4. `compact_support_complex` gives $C_c^{\bullet}(\Pi; \mathcal{G})$.

4 Examples and usage

This section provides sample code and output for some specific examples. These examples are chosen so as to highlight the connections to cohomology of complex algebraic varieties described in Section 2.2.

4.1 Polytopes

We consider the polyhedral complex that consists of a three dimensional cube C and all its faces. We will compute the W -sheaves for C as well as the Betti numbers of the cohomology groups $H^q(C; W^p)$ for all p, q from 0 to 3.

```
application "fan";
$pc = new PolyhedralComplex(
    check_fan_objects(new Cone(cube(3))));
```

```

@beti = ();
for(my $i=0; $i<4; $i++){
  my $w = $pc->wsheaf($i);
  my $s = $pc->usual_cochain_complex($w);
  push @beti, $s->BETTI_NUMBERS;
}
print new Matrix(@beti);

```

The first step turns the three dimensional cube into a polyhedral complex. Then we loop over all possible W -sheaves and save the Betti numbers in a matrix. This results in

```

fan > print new Matrix(@beti);
1 0 0 0
0 3 0 0
0 0 3 0
0 0 0 1

```

We see that $\dim H^q(C; W^p) = 0$ if $p \neq q$. The diagonal $\dim H^p(C; W^p)$ is the dual h -vector of the polytope that defines the polyhedral complex. This relationship holds for any simple polytope Δ . See [3, Corollary, pg. 6] for the statement in terms of the dual fan of Δ .

```

fan > $cube = polytope::cube(3);

```

```

fan > print $cube->DUAL_H_VECTOR;
1 3 3 1

```

The toric variety of the three dimensional cube is $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Notice that $\dim H^p(X; \Omega^p) = 1$ if $p = 0, 3$, $\dim H^p(X; \Omega^p) = 3$ if $p = 1, 2$ and $\dim H^q(X; \Omega^p) = 0$ for $p \neq q$.

4.2 Bergman fans and tropical linear spaces

We can build the Bergman fan $B(M)$ of a matroid M and compute the usual homology of the F -cosheaf - this means that we only consider bounded faces. Assuming the matroid is connected, the Bergman fan of a matroid has a unique bounded face which is the vertex v . Therefore, the cellular chain groups $C_q(B(M); F_p)$ are 0 unless $q = 0$.

In the following examples we will see that

$$\dim H_0(B(M); F_p) = \dim OS^p(M),$$

where $OS^p(M)$ is the p^{th} graded part of the Orlik-Solomon algebra of M . This follows from Theorem 2. Notice that when v is the vertex of the Bergman fan,

$$H_0(B(M); F_p) = C_0(B(M); F_p) = F_p(v)$$

and $H_i(B(M); F_p) = 0$ for $i > 0$.

When M is a rank $d+1$ matroid on $n+1$ elements arising from a non-central hyperplane arrangement \mathcal{A} in $\mathbb{C}P^d$, the Orlik-Solomon algebra is isomorphic to the cohomology ring of the complement $\mathcal{C} := \mathbb{C}P^d \setminus \mathcal{A}$ of the arrangement. There is a canonical embedding of $\mathcal{C} \rightarrow (\mathbb{C}^*)^n$, and the tropicalisation of this is the Bergman fan of the matroid. Therefore, we see that the homology of the F -cosheaf on a tropicalisation recovers cohomological information about the original variety.

Example 4. Our first example is to compute the tropical homology of a tropical line in \mathbb{R}^2 . This is the tropicalisation of a generic line $L \subset \mathbb{C}^2$ intersected with the torus $(\mathbb{C}^*)^2$. Notice that this space is homeomorphic to $\mathbb{C}P^1 \setminus \{p_1, p_2, p_3\}$, so that $\dim H^0(L \cap (\mathbb{C}^*)^2; \mathbb{C}) = 1$ and $\dim H^1(L \cap (\mathbb{C}^*)^2; \mathbb{C}) = 2$. The tropical line is the Bergman fan of the uniform matroid of rank 2 on 3 elements.

We start by creating this polyhedral complex in *polymake*:

```
application "matroid";
$m = uniform_matroid(2,3);
application "tropical";
$t = matroid_fan<Max>($m);
$t->VERTICES;
application "fan";
$berg = new PolyhedralComplex($t);
```

Next, we construct the associated F -cosheaves up to the dimension of the Bergman fan and compute their usual chain complexes:

```
$f0 = $berg->fcosheaf(0);
$f1 = $berg->fcosheaf(1);
$s0 = $berg->usual_chain_complex($f0);
$s1 = $berg->usual_chain_complex($f1);
```

Now we ask for the Betti numbers and obtain:

```
fan > print $s0->BETTI_NUMBERS;
1 0
fan > print $s1->BETTI_NUMBERS;
2 0
```

We can also compute the Borel-Moore homology. Here every face of the Bergman fan contributes to the Borel-Moore chain groups (see Definition 5).

```
$bm0 = $berg->borel_moore_complex($f0);
$bm1 = $berg->borel_moore_complex($f1);
```

gives

```
fan > print $bm0->BETTI_NUMBERS;
0 2
fan > print $bm1->BETTI_NUMBERS;
0 1
```

Notice that we obtain $\dim H_q(B(M); F_p) = \dim H_{d-q}^{BM}(B(M); F_{d-p})$ for $d = 1$, which is the dimension of the Bergman fan. This is the homological version of Poincaré duality for matroidal fans and tropical manifolds from [16].

Example 5. In this example we will study the Bergman fan of the matroid of the complete graph on four vertices. This is the matroid of the so-called braid arrangement of lines in \mathbb{CP}^2 , whose complement is the moduli space of 5-marked genus 0 curves $\mathcal{M}_{0,5}$ (see [1]). We use the applications “graph” and “matroid” to first construct the Bergman fan.

```
application "graph";
$g = complete(4);
application "matroid";
$m = matroid_from_graph($g);
application "tropical";
$t = matroid_fan<Max>($m);
$t->VERTICES;
application "fan";
$berg = new PolyhedralComplex($t);
```

We compute the usual and the Borel-Moore homology of the F -cosheaf.

```
@betti_usual = ();
@betti_bm = ();
for(my $i=0; $i<3; $i++){
    my $f = $berg->fcosheaf($i);
    my $s = $berg->usual_chain_complex($f);
    my $bm = $berg->borel_moore_complex($f);
    push @betti_usual, $s->BETTI_NUMBERS;
    push @betti_bm, $bm->BETTI_NUMBERS;
}
```

This gives the following Betti numbers:

```
fan > print new Matrix(@betti_usual);
1 0 0
5 0 0
6 0 0

fan > print new Matrix(@betti_bm);
0 0 6
0 0 5
0 0 1
```

Again we see that we have $\dim H_q(B(M); F_p) = \dim H_{d-q}^{BM}(B(M); F_{d-p})$, where now $d = 2$.

Example 6. A tropical linear space is not necessarily a fan. Nevertheless the Betti numbers of the tropical homology of the tropical linear space and of its recession fan agree. In this example, we start with the Bergman fan of the uniform matroid of rank 3 on 6 elements and compare its homology with that of the tropical linear space of a valuated matroid with the aforementioned matroid as its underlying matroid.

```
$m = matroid::uniform_matroid(3,6);
$t = tropical::matroid_fan<Max>($m);
$t->VERTICES;
application "fan";
$berg = new PolyhedralComplex($t);
@betti_usual = ();
@betti_bm = ();
for(my $i=0; $i<3; $i++){
    my $f = $berg->fcosheaf($i);
    my $s = $berg->usual_chain_complex($f);
    my $bm = $berg->borel_moore_complex($f);
    push @betti_usual, $s->BETTI_NUMBERS;
    push @betti_bm, $bm->BETTI_NUMBERS;
}
```

gives

```
fan > print new Matrix(@betti_usual);
1 0 0
5 0 0
10 0 0

fan > print new Matrix(@betti_bm);
0 0 10
0 0 5
0 0 1
```

Now we consider a valuated matroid whose underlying matroid is uniform of rank 3 on 6 elements and construct the corresponding tropical linear space.

```
$v = [0,0,3,1,2,1,0,1,0,2,2,0,3,0,4,1,2,2,0,0];
$val_matroid = new matroid::ValuatedMatroid<Min>(
    BASES=>matroid::uniform_matroid(3,6)->BASES,
    VALUATION_ON_BASES=>$v,N_ELEMENTS=>6);
$tls = tropical::linear_space($val_matroid);
@betti_usual = ();
@betti_bm = ();
for(my $i=0;$i<3;$i++){
    my $fi = $tls->fcosheaf($i);
```

```

    my $si=$tls->usual_chain_complex($fi);
    my $bmi=$tls->borel_moore_complex($fi);
    push @betti_usual, $si->BETTI_NUMBERS;
    push @betti_bm, $bmi->BETTI_NUMBERS;
}

```

returns,

```

fan > print new Matrix(@betti_usual);
1 0 0
5 0 0
10 0 0

```

```

fan > print new Matrix(@betti_bm);
0 0 10
0 0 5
0 0 1

```

which is the same as for the Bergman fan of the matroid above.

Example 7. This example demonstrates that the usual cohomology and the compactly supported cohomology of the W -sheaves on tropical linear spaces seem to satisfy some interesting and also potentially useful vanishing theorems. We demonstrate this with a single example and then summarise the vanishing phenomena in Conjecture 1 below. We continue with the same tropical linear space from Example 6.

```

@wbetti_usual = ();
@wbetti_cs = ();
for(my $i=0;$i<3;$i++){
    my $wi = $tls->wsheaf($i);
    my $wsi=$tls->usual_cochain_complex($wi);
    my $wcsi=$tls->compact_support_complex($wi);
    push @wbetti_usual, $wsi->BETTI_NUMBERS;
    push @wbetti_cs, $wcsi->BETTI_NUMBERS;
}

```

returns,

```

fan > print new Matrix(@wbetti_usual);
1 0 0
0 4 0
0 0 1

```

```

fan > print new Matrix(@wbetti_cs);
0 0 10

```

0 0 32
0 0 28

Conjecture 1. *Let $L \subset \mathbb{R}^n$ be a tropical linear space of dimension d . Then we have*

$$H^q(L; W^p) = 0 \quad \text{if } p \neq q \quad \text{and} \quad H_c^q(L; W^p) = 0 \quad \text{if } q \neq d.$$

To date, the above conjecture has been checked on all realisable tropical linear spaces in $\text{Trop}(\text{Gr}(3, 6))$ using our package.

By considering the Euler characteristics of the complexes $C^\bullet(L; W^p)$ and $C_c^\bullet(L; W^p)$ we have:

$$(-1)^p H^p(L; W^p) = \sum_{q=0}^d (-1)^q \binom{q}{p} f_q^b,$$

$$(-1)^d H_c^d(L; W^p) = \sum_{q=0}^d (-1)^q \binom{q}{p} f_q$$

where $f^b = (f_0^b, \dots, f_d^b)$ is the f -vector of the bounded faces of L and $f = (f_0, \dots, f_d)$ is the f -vector of L . If the above conjecture holds, then understanding the f -vector of a tropical linear space comes down to understanding the possible dimensions of $H_\bullet^q(L; W^p)$. For example, it is possible to bound the f^b -vector by bounding $H^p(L; W^p)$. This would give an approach to the f -vector conjecture for tropical linear spaces (see [23]) similar to the proof of the upper bound conjecture for polytopes.

4.3 Tropical hypersurfaces

Using the `a-tint` package ([12]) we can construct tropical hypersurfaces in *polymake* from piecewise integer affine functions which are convex, i.e. tropical polynomials. These examples demonstrate how one can start directly with a given tropical polynomial and compute the homology of the F -sheaves on the tropical hypersurface. In other words, the tropical homology of the tropical hypersurface.

Example 8. We begin with a tropical curve in \mathbb{R}^2 which is dual to a triangulation of a square of size 1.

```
application "tropical";
$f = toTropicalPolynomial("max(0,x+5,y+3, x+y+9)");
$div = divisor( (projective_torus<Max>(2)),
  rational_fct_from_affine_numerator($f));
application "fan";
@betti_usual = ();
@betti_bm = ();
for(my $i=0;$i<2;$i++){
  my $fi = $div->fcosheaf($i);
```

```

    my $si=$div->usual_chain_complex($fi);
    my $bmi=$div->borel_moore_complex($fi);
    push @betti_usual, $si->BETTI_NUMBERS;
    push @betti_bm, $bmi->BETTI_NUMBERS;
}

```

gives

```

fan > print new Matrix(@betti_usual);
1 0
3 0

```

```

fan > print new Matrix(@betti_bm);
0 3
0 1

```

Example 9. As a final example, we calculate the homology of another tropical hypersurface. This hypersurface arises as a triangulation of the three dimensional simplex of edge length 4, and is a tropical K3-surface in \mathbb{R}^3 .

```

application "tropical";
$f = toTropicalPolynomial("max(0,x,y,z, 2*x-2,
    2*y-2, 2*z-2, x+y-1, x+z-1, y+z-1, 3*x-6,
    3*y-6, 3*z-6, 2*x+y-4, 2*y+x-4, 2*x+z-4,
    2*z+x-4, 2*y+z-4, 2*z+y-4, x+y+z+1, 4*x-12,
    4*y-12, 4*z-12, 3*x+y-9, 3*y+x-9, 3*x+z-9,
    3*z+x-9, 3*y+z-9, 3*z+y-9, 2*x+2*y-8,
    2*x+2*z-8, 2*y+2*z-8, 2*x+y+z-7, x+2*z+y-7,
    2*y+z+x-7)");
$k3 = divisor((projective_torus<Max>(3)),
    rational_fct_from_affine_numerator($f));
application "fan";
@numbers = (0..2);
@cosheaves = map{$k3->fcosheaf($_)} @numbers;
@usualChainComplexes = map{$k3->usual_chain_complex($_)}
    @cosheaves;
@bmComplexes = map{$k3->borel_moore_complex($_)}
    @cosheaves;
@betti_usual = map{$_->BETTI_NUMBERS}
    @usualChainComplexes;
@betti_bm = map{$_->BETTI_NUMBERS} @bmComplexes;

```

We obtain the following matrices of Betti numbers:

```

fan > print new Matrix(@betti_usual);

```



```

1 0 1
3 31 0
34 0 0

```

```

fan > print new Matrix(@beti_bm);
0 0 34
0 31 3
1 0 1

```

This tropical hypersurface is bigger than the polyhedral complexes we considered before. Its f -vector is $(64, 96, 34)$. This can be seen from the usual chain complex of the F^0 -cosheaf.

```

fan > $usualChainComplexes[0]->print();
3      2      1      0      -1
k^0 --> k^34 --> k^96 --> k^64 --> k^0

```

In this example and Example 8, we again observed the homological version of Poincaré duality.

5 Future directions

5.1 Sheaves of modules

It is also possible to compute (co)homology of cellular (co)sheaves of modules. For example, given a rational polyhedral complex there are also integral versions of the W -sheaves and F -cosheaves, which are free \mathbb{Z} -modules. However, using the current methods `fcosheaf` and `wsheaf` can lead to incorrect results over \mathbb{Z} . Still, the ranks of the torsion and the free part of the (co)homology will be correct in these cases.

The problem with using the current implementation to compute integral versions of the (co)homology of the integral versions of these (co)sheaves is that the property **BASES** does not necessarily consist of a lattice basis of the free \mathbb{Z} -module for each face. In addition, the matrices in **BLOCKS** may not accurately encode the \mathbb{Z} -linear maps.

If one properly chooses \mathbb{Z} -bases for **BASES** and defines **BLOCKS** manually with the correct maps over \mathbb{Z} when creating a (co)sheaf, then the current rules for computing the cellular (co)homology will compute the correct \mathbb{Z} -homology.

We plan to adapt `fcosheaf` and `wsheaf` to give the correct results over \mathbb{Z} after switching to *polymake*'s internal chain complex object. This has recently been pushed to the *polymake* repository by Olivia Röhrig.

5.2 Tropical compactifications and projective hypersurfaces

How to implement compact tropical varieties is part of an ongoing discussion inside the *polymake* developer team. One possibility is to save one affine tropical variety per chart of the tropical projective space. For many cases this would result in a drastic increase

of resource usage. Thus, one may want to restrict to certain classes of tropical varieties with nice compactifications.

A solution to this problem is necessary in order to use our extension in order to give answers for example Problem 10 on Surfaces in [24]. Upon having an implementation of compact tropical varieties, one could for example combine our package and the approach to tropical Enriques surfaces in [2] to determine the Hodge numbers and solve Problem 10.

5.3 Implementing other cellular (co)sheaves

There are many other imaginable cellular (co)sheaves to consider on a polyhedral complex, including (co)sheaves arising from common (co)sheaf operations, such as restrictions, pullbacks, and Verdier duals.

Cellular sheaves on polyhedral fans have appeared in the work of Brion (see [3]). There, given a polyhedral fan in \mathbb{R}^n , one associates to a face σ , the vector space $\mathbb{R}^n / \text{Lin}(\sigma)$, where $\text{Lin}(\sigma)$ denotes the linear span of the σ . These vector spaces come equipped with natural maps between them when there is an inclusion of faces. One can also take p^{th} exterior powers of these vector spaces, as well as generalise this definition beyond polyhedral fans to get a collection of sheaves.

The cohomology of these cellular sheaves is related to the motion spaces in discrete dynamical geometry (see [25]). Following the descriptions in that paper, the W -sheaves come up in aspects of rigidity and the F -cosheaves of skeleta of polyhedral complexes are related to stress spaces.

5.4 Applied topology

Celluar (co)sheaves have also appeared often in the field of applied topology and topological data analysis, notably in the study of sensor networks, network coding, and persistent homology (see [5], [11]). Although the (co)sheaves appearing in these contexts are often a part of the input data of the model under consideration and do not have a simple recipe coming from the geometry of the underlying topological space like in the case of tropical homology. However, our extension allows for the construction of a sheaf from scratch. Another generalisation to be considered in the future, is that the underlying topological spaces appearing in this context are not necessarily polyhedral complexes in \mathbb{R}^n . The current extension capabilities for (co)sheaves on polyhedral complexes could be extended to more general topological spaces using the *polymake* application *topaz*.

We would also like to point out the current efforts underway by Olivia Röhrig to implement persistent homology in *polymake*.

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References

- [1] Federico Ardila and Carly J. Klivans: The Bergman complex of a matroid and phylogenetic trees. *J. Comb. Theory Ser. B*, 96(1):38–49, 2006.
- [2] Barbara Bolognese, Corey Harris, and Joachim Jelisiejew: Equations and tropicalization of Enriques surfaces. *op. cit.*
- [3] Michel Brion. The structure of the polytope algebra. *Tohoku Math. J. (2)*. (1997) Vol. 49, No.1, 1–32.
- [4] Erwan Brugallé, Ilia Itenberg, Grigory Mikhalkin, and Kristin Shaw: Brief introduction to tropical geometry. To appear in *Proceedings of Gökova Geometry and Topology 2014*. Preprint at: <http://arxiv.org/abs/1502.05950>.
- [5] Justin Curry: Sheaves, cosheaves and applications. <https://arxiv.org/abs/1303.3255>.
- [6] Vladimir I. Danilov, Askold G. Khovanskii: Newton polyhedra and an algorithm for computing Hodge–Deligne numbers. *Izv. Akad. Nauk SSSR Ser. Mat.* (1986) Vol. 50, Issue 5, 925–945
- [7] Vladimir I. Danilov: The geometry of toric varieties. *Russian Math. Surveys*. 33:2 (1978) 97–154.
- [8] Wolfram Decker, Gert-Martin Greuel, Gerhard Pfister, Hans Schönemann: SINGULAR 4-0-2 — A computer algebra system for polynomial computations. <http://www.singular.uni-kl.de> (2015).
- [9] William Fulton: *Introduction to toric varieties*, volume 131 of *Ann. Math. Studies*. Princeton Univ. Press., 1993.
- [10] Evgenij Gawrilow and Michael Joswig: polymake: a framework for analyzing convex polytopes. *Polytopes – combinatorics and computation (Oberwolfach, 1997)*, 43 – 73, DMV Sem., 29, Birkhäuser, Basel, 2000. MR1785292
- [11] Robert Ghrist and Yasuaki Hiraoka: Applications of sheaf cohomology and exact sequences to network coding. *Proc. NOLTA*, 2011.
- [12] Simon Hampe. **a-tint**: a polymake extension for algorithmic tropical intersection theory: *European J. Combin.* (2014) Vol. 36, 579–607.

- [13] Simon Hampe and Michael Joswig: Tropical Computations in polymake. arXiv:1612.02581 (2016)
- [14] Sir William V.D. Hodge: *The Theory and Applications of Harmonic Integrals*. Cambridge mathematical library (1941)
- [15] Ilia Itenberg, Ludmil Katzarkov, Grigory Mikhalkin, Ilia Zharkov: Tropical Homology. arXiv:1604.01838 (2016)
- [16] Philipp Jell, Krisin Shaw, Jascha Smacka: Superforms, tropical cohomology and Poincaré duality. arXiv:1512.07409 (2016).
- [17] Anders Jensen: Gfan, a software system for Gröbner fans and tropical varieties. Available at <http://home.imf.au.dk/jensen/software/gfan/gfan.html>
- [18] Eric Katz: Matroid theory for algebraic geometers. <http://arxiv.org/abs/1409.3503>.
- [19] Eric Katz and Alan Stapledon: Tropical geometry, the motivic nearby fiber, and limit mixed Hodge numbers of hypersurfaces. Research in the Mathematical Sciences. (2016) Vol. 3, Issue 1, 1–36.
- [20] Diane Maclagan and Bernd Sturmfels: *Introduction to tropical geometry*. Providence, RI: American Mathematical Society (AMS), 2015.
- [21] Peter Orlik and Hiroaki Terao: *Arrangements of hyperplanes*. Springer Verlag, 1992.
- [22] James Oxley: *Matroid theory*, volume 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, second edition, 2011.
- [23] David Speyer: Tropical linear spaces. SIAM J. Discrete Math. (2008) Vol. 22, No. 4, 1527–1558.
- [24] Bernd Sturmfels: Fitness, Apprenticeship, and Polynomials. arXiv:1612.03539.
- [25] Tiong-Seng Tay and Walter Whiteley: A homological interpretation of skeletal rigidity. Adv. in Appl. Math. (2000) Vol 25, No.1, 102–151.
- [26] Ilia Zharkov: The Orlik-Solomon algebra and the Bergman fan of a matroid. J. Gökova Geom. Topol. GGT. (2013) Vol 7, 25–31.

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